A metabolic limit on the ability to make up for lost time in endurance events

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**Fukuba, Yoshiyuki, and Brian J. Whipp.** A metabolic limit on the ability to make up for lost time in endurance events, J. Appl. Physiol. 87(2): 853–861, 1999.—It has been repeatedly demonstrated that the tolerable duration (t) of high-intensity cycling is well characterized as a hyperbolic function of power (P) with an asymptote that has been termed the “fatigue threshold” and with a curvature constant. This hyperbolic P-t relationship has also been confirmed in running and swimming, when speed (V) is used instead of P; that is, \((V - V_F)t = D_\prime\), where \(V_F\) is the \(V\) at the fatigue threshold, and \(D_\prime\) is the curvature constant. Therefore, we theoretically analyzed herein the consequences of an athlete performing the initial part of an endurance event at a \(V\) different from the constant rate that would allow the performance time to be determined by the hyperbolic V-t relationship. We considered not only the V-t constraints that limit the athlete’s ability to make up the time lost by too slow an early pace but also the consequences of a more rapid early pace. Our analysis demonstrates that both the \(V_F\) and \(D_\prime\) parameters of the athlete’s V-t curve play an important role in the pace allocation strategy of the athlete. That is, 1) when the running \(V\) during any part of the whole running distance is below \(V_F\), the athlete can never attain the goal of achieving the time equivalent to that of running the entire race at constant maximal \(V\) (i.e., that determined by one’s own best V-t curve); and 2) the “endurance parameter ratio” \(D_\prime/V_F\) is especially important in determining the flexibility of the race pace that the athlete was able to choose intentionally.

power-duration hyperbolic curve; speed-duration hyperbolic curve; endurance athlete performance; race-pace strategy; theoretical analysis

THE CHOICE OF SPEED, and the variations of speed, that will maximize an endurance athlete’s ability to succeed in winning a race involves a complex interplay of physiological and psychological factors. An inappropriate allocation of pace can reduce the athlete’s likelihood of success in the event. One aspect of the physiological basis of race-pace strategy has not, we believe, received appropriate consideration.

This is, perhaps, best exemplified by the physiological inferences that may be drawn from the answer to the following question. Imagine that two athletes engage in a race of some specified distance X. Athlete A runs the race at a constant speed; athlete B runs the first one-half of the race at one-half the speed of athlete A. At what speed must athlete B run the second one-half of this race to finish at the same time as athlete A?

The answer, of course, is that it is impossible: athlete B cannot possibly catch athlete A. The reason is simply that, in running one-half the distance at one-half the speed, athlete B has used all of the winning time of athlete A. By extension, athletes may use the strategy of relatively slow running early in the race, either in an attempt to husband their reserves for a subsequent sprint or as a tactic to induce suboptimal performance in other athletes. That is, the strategy requires proportionately higher subsequent velocities to achieve the same effective time that would have resulted from constant-speed running. This increases the likelihood of draining the limited and, it has been proposed, constant metabolic energy reserve that is available to the athlete.

The tolerable duration of a particular work rate has been shown to be an inverse function of the work rate, as shown in Fig. 1. It has further been demonstrated (10, 11, 14, 15, 20) that this relationship is hyperbolic, with an asymptote on the power axis that has been termed “the critical power” (CP) or “fatigue” threshold (\(V_F\)) and a curvature constant W that, being the product of power and time, has the units of work, i.e., equivalent to a constant amount of energy above CP. This constant amount of energy may notionally be utilized rapidly by exercising at high power outputs, or may be eked out for longer durations by exercising at lower work rates.

It is important to emphasize, first, that the hyperbolic relationship is unlikely to provide a precise representation of the actual physiological behavior at the very extremes of performance, because of distorting factors such as 1) limitations of mechanical force generation in muscle for the very highest power or running speed and 2) constraints resulting from substrate provision and thermoregulatory or body fluid requirements for markedly prolonged exercise (21). The duration limit for the application of the hyperbolic relationships has not been precisely established, but it does appear to be applicable up to ~20 or 30 min.

We consider here the consequences of an athlete performing the initial part of an endurance event at a speed different from that of the constant rate that would allow the performance time to be determined by the power-duration relationship. In this context, by
endurance we mean those events for which deliberate pace changes might be a plausible competitive strategy; i.e., sprints are not considered. We consider not only the power-duration constraints that limit the athlete’s ability to make up the time lost by keeping too slow an early pace but also the consequences of keeping a more rapid early pace. In other words, we consider the pace allocation to achieve the best possible time for a runner. Our results demonstrate that the threshold and curvature parameters of the subject’s power-duration curve are likely to be important in establishing the limits of the pace allocation strategy of the athlete.

**THE HYPERBOLIC POWER-DURATION RELATIONSHIP FOR CYCLING**

It is well known that the relationship between power (P; W or J/s) and its endurance time (t; s) for high-intensity cycling may be characterized as a hyperbolic function (e.g., see Ref. 2 for review). That is

\[ (P - \theta_f) \cdot t = W' \]  \hspace{1cm} (1a)

or

\[ P = W' \cdot (1/t) + \theta_f \]  \hspace{1cm} (1b)

where \( \theta_f \) (W) is the power asymptote for the P-t hyperbola and consequently represents a unique power above which the time to fatigue is predictably determined (11, 14). \( W' \) represents a constant amount of work that can be performed above \( \theta_f \). It has been considered to represent the energy store components: \( O_2 \) stores, phosphagen, and anaerobic glycolysis (22).

The total amount of work done (\( W_{tot} \)), however, is

\[ W_{tot} = P \cdot t \]  \hspace{1cm} (2)

Therefore, Eq. 1b is rewritten as

\[ W_{tot} = \theta_f \cdot t + W' \]  \hspace{1cm} (3)

where \( W_{tot} > W' \). In the case of the subject in Fig. 1, for example, \( \theta_f \) and \( W' \) are 168 W and 15.6 kJ, respectively.

**THE HYPERBOLIC VELOCITY-DURATION RELATIONSHIP FOR RUNNING**

Whereas the hyperbolic P-t relationship has been confirmed by several previous studies for cycling performance (11, 14, 15, 20), it has also been confirmed to be hyperbolic in running and swimming, when speed or velocity (V) is used instead of P (3–5, 12, 18). In the following discussion, we therefore use the term V instead of P. That is

\[ (V - V_F) \cdot t = D' \]  \hspace{1cm} (4a)

or

\[ V = D' \cdot (1/t) + V_F \]  \hspace{1cm} (4b)

where \( V_F \) is now the speed asymptote (m/s), and distance (D’) is expressed in meters. As the total running distance \( [D (m)] \) is

\[ D = V \cdot t \]  \hspace{1cm} (5)

Eq. 4 may be rewritten as

\[ D = V_F \cdot t + D' \]  \hspace{1cm} (6)

The relationships among Eqs. 4a, 4b, and 6 are graphically represented in Fig. 2, A, B, and C, respectively.

**MAXIMAL AVERAGE SPEED**

Each individual is assumed to have a specific hyperbolic relationship between \( V \) and \( t \). Each is therefore able to run some specific distance \( X \) at the maximal average speed \( [V_{max(X)}]\), which is determined by the crossing point of the particular \( V-t \) curve defined by Eqs. 4a, 4b, or 6 (i.e., the “best \( V-t \) curve to \( D' \)”) and the \( V-t \) relationship curve to some specific distance \( X \) defined by Eq. 5 (i.e., the “distance curve to \( X \)”). That is, Eq. 5 with \( X \) substituted for \( D \) puts in Eq. 4b

\[ V_{max(X)} = V_F/[1 - (D'/X)] \]  \hspace{1cm} (7)

where \( X > D' \).
The running time $t_{\max(x)}$ to distance $X$ run with a constant speed $V_{\max(x)}$ is

$$t_{\max(x)} = X/V_{\max(x)} = (X - D)/V_F \quad (8)$$

As a simple example, consider the $V_{\max(x)}$ and $t_{\max(x)}$ of some subject whose $V_F$ and $D'$ are 4.4 m/s and 300 m, respectively, i.e., when the subject runs at the constant $V_{\max(x)}$ throughout the 5,000-m distance. According to Eqs. 7 and 8, $V_{\max(x)}$ and $t_{\max(x)}$ of this subject for a 5,000-m run are 4.68 m/s and 1,068 s (17 min 48 s), respectively. Figure 2, A-C, provides the graphic representations of this example, i.e., the crossing point between both curves derived from Eqs. 4a, 4b, or 6, and the line from $A-C$, provides the detail.

### ALLOCATION OF RUNNING PACE

In general, runners change the running speed during a race according to their own pace-strategy, although other runners’ tactics can naturally alter this. We shall consider the simple situation in which the runner runs at different speeds ($V_1$ and $V_2$) during the parts of divided distances ($X_1$ and $X_2$) of total distance $X$.

During the initial part ($X_1$), the runner is only able to choose a speed ($V_1$) below $V_{\max(x)}$, which is determined as the crossing point of the V-t curve to $D'$ and the distance curve to $X_1$. If $V_1$ is above the $V_F$ of the runner, the maximum speed ($V_2$), which the runner can choose during the second part of the race ($X_2$), is automatically determined by the amount of $D'$ remaining [i.e., $D'(x_2)$], because a specific amount of $D'$ [D'($x_1$)] is already expended during $X_1$. However, if $V_1$ is below $V_F$, the runner can subsequently run at a speed ($V_2$) which utilizes the entire $D'$ to the limit of the best V-t curve. Note that, in this case, the runner cannot run above the speed $[V_{\max(x_2)}]$, which is determined by the best V-t curve to $D'(x_2)$ and distance curve to $X_2$. In the following discussion, therefore, we consider $V_1$ in greater detail.

$V_F < V_1 < V_{\max(x_1)}$

In the initial part $X_1$, the runner uses the $D'(x_1)$ and the time ($t_1$) necessary to run distance $X_1$ at a speed $V_1$, which is

$$t_1 = X_1/V_1 = [X_1 - D'(x_1)]/V_F \quad (9)$$

where $0 \leq D'(x_1) \leq D'$ and $D'(x_1) < X_1$. In the second part $X_2$, $V_2$ is consequently determined by the remaining $D'$ [$D'(x_2)$]. Therefore, the time ($t_2$) that is taken to run $X_2$ at a speed $V_2$ [$V_F < V_2 < V_{\max(x_2)}$] is

$$t_2 = X_2/V_2 = [X_2 - D'(x_2)]/V_F \quad (10)$$

where $D'(x_2) < X_2$. If the runner runs the total distance $X$ with this pace allocation, then the total time for the race ($t_{tot}$) is

$$t_{tot} = t_1 + t_2 = [X_1 - D'(x_1)]/V_F + [X_2 - D'(x_2)]/V_F$$

$$= (1/V_F)\cdot\left([X_1 + X_2] - [D'(x_1) + D'(x_2)]\right) = (X - D')/V_F \quad (11)$$

That is, $t_{tot}$ is exactly same as the time $[t_{\max(x)}]$ when the runner runs at the constant speed $V_{\max(x)}$, i.e., the speed determined by the best V-t curve to $X$ and distance curve to $X$.

$V_1 < V_F$

In the initial part $X_1$, if the runner chooses a speed ($V_1$) that is below $V_F$, the time ($t_1$) that is taken to run distance $X_1$ with $V_1$ ($V_1 < V_F$) is

$$t_1 = X_1/V_1 \quad (12)$$

In the second part $X_2$, the runner can now utilize the entire $D'$ as $D'(x_2)$, at the speed $V_2$ determined by the best V-t curve to $D'$ and the distance curve to $X_2$. The $t_2$ [$= t_{\max(x_2)}$], which is taken for $X_2$ with $V_2$ [$= V_{\max(x_2)}$], is
\[ t_2 = \frac{X_2}{V_2} = \frac{(X_2 - D')/V_F}{[X_1 + \frac{X_2}{V_F}]} \]

where \( D' < X_2 \). Therefore, total running time \( t_{\text{tot}}^* \)

\[ t_{\text{tot}}^* = t_1 + t_2 = \frac{(X_1/V_1) + [(X_1 - X_2)/V_F]}{[X_1 + \frac{X_2}{V_F}]} \]

In a comparison of \( t_{\text{tot}}^* \) to the time \( t_{\text{max}}^* \), when the runner runs at the constant \( V_{\text{max}}(X) \) throughout the whole distance \( X \), the \( t_{\text{tot}}^* \) is always longer than \( t_{\text{max}}^* \), where \( X_1 > 0 \), because the second term of Eq. 14 is equal to \( t_{\text{max}}^* \).

As a consequence, when the runner runs at \( V_1 \) below \( V_F \), it is not possible to make up for the lost time during the second part of the race (\( X_2 \)). That is, even if the runner runs the \( X_2 \) distance with \( V_F \) at \( V_{\text{max}}(X_2) \), the athlete can never attain the goal of achieving the time equivalent to that of running the entire race at the average speed \( V_{\text{max}}(X) \), i.e., that determined by his or her own best \( V \)-t curve to \( D' \) and distance curve to \( X \). A more general theory of the pace allocation problem is presented in the APPENDIX.

**NUMERICAL AND GRAPHIC EXAMPLES**

Here we consider the 5,000 m by using the same subject from **MAXIMAL AVERAGE SPEED** from a numerical and graphic standpoint. We choose the subject’s \( V_F \) and \( D' \) to be 4.4 (m/s) and 300 (m), respectively, and the \( V_{\text{max}} \) and \( t_{\text{max}} \) for this race are 4.68 (m/s) and 1,068.2 (s), respectively (see Fig. 2A).

We used one of a number of plausible allocations of running pace in \( V_F \leq V_1 < V_{\text{max}}(X_1) \). The initial part \( X_1 \) is 3,500 m with a speed \( (V_1) \), which is relatively slow but still above \( V_F \), e.g., 4.50 (m/s). This speed corresponds to 3.6% below \( V_{\text{max}} \) at 5,000 m \( (V_{\text{max}}(5,000)) \). For the initial 3,500 m, \( t_1 \) and \( D'(3,500) \) are 777.8 (s) and 77.8 (m), respectively. These values are determined by the crossing point of the runner’s \( V \)-t curve to \( D'(3,500) (=77.8 \text{ m}) \) and the distance curve for 3,500 m (Fig. 3A). For the second 1,500 m, because the runner has access only to the remaining \( D'(1,500) (=300-77.8=222.2 \text{ m}) \) for the last “spurt” (i.e., “remaining energy”), the athlete can run only at the speed that is determined by the crossing point of the best \( V \)-t curve to \( D'(1,500) (=222.2 \text{ m}) \) and the distance curve to 1,500 m (Fig. 3B). This requires the second 1,500 m to be run in a \( t_2 \) of 290.4 (s) (i.e., “required time”) and at a speed \( V_2 \) of 5.165 (m/s). This is +10.3% greater than \( V_{\text{max}}(5,000) \). As a result, \( t_{\text{tot}} \), i.e., the race record, is 1,068.2 (s), and is exactly the same to \( t_{\text{max}} \), which is derived from maximal average speed, \( V_{\text{max}}(5,000) \).

Consider, however, if the runner chooses the speed during the initial part that is below \( V_F \), e.g., 4.21 (m/s) as in \( V_1 < V_F \). This corresponds to 10.0% below \( V_{\text{max}}(5,000) \) for the initial 3,500 m of the 5,000-m distance. Throughout \( X_1 \), the runner can reserve the whole \( D' \) for the last spurt (i.e., remaining energy). Note, however, that 831.4 (s) is already taken as \( t_1 \) (Fig. 3C). This requires the second 1,500 m to be run in a \( t_2 \) of 236.8 (s) (i.e., required time). In the second 1,500 m, the athlete can run at the maximal speed \( V_{\text{max}}(1,500) \), which is determined by the crossing point between the runner’s best \( V \)-t curve to \( D'(=300 \text{ m}) \) and the distance curve to 1,500 m. Throughout the second part, \( t_2 \) is 272.7 (s) and \( V_2 \) is 5.501 (m/s), which is about +25.0% greater than \( V_{\text{max}}(5,000) \) (Fig. 3D). As a result, \( t_{\text{tot}} \) is 1,104.1 (s); this is some 36 s longer than \( t_{\text{max}} \). This is the case for all durations of suboptimal early race pace that are below \( V_F \).

**GENERAL CONSIDERATIONS**

Here we simulate these relationships systematically for the two-division pace-allocation strategy in the 5,000 m. Furthermore, we also examine the effect of \( V_F \) and \( D' \) on this strategy. We used the actual values for the runner determined from the study of Hughson et al. (Fig. 3 in Ref. 4); i.e., \( V_F \) and \( D' \) are ~5.0 (m/s) and ~150 (m), respectively.

Consider the expected race performance when the athlete runs a 5,000-m distance at a maximal average speed, i.e., as determined by the \( V \)-t hyperbola: in this case \( V_{\text{max}} \) is 5.155 (m/s) and \( t_{\text{tot}} \) is 970 (s). We examined the systematic effect of pace allocation by utilizing two different speeds (\( V_1, V_2 \)) in different combinations of \( X_1-X_2 \) to the total 5,000 m. \( X_1 \) and \( X_2 \) (m) were chosen to be 1,000-4,000, 2,000-3,000, 3,000-2,000, and 4,000-1,000. In each \( X_1-X_2 \) combination, we calculated the parameters according to Eqs. 9-14. \( V_1 \) was varied systematically, from ~20% of \( V_{\text{max}}(5,000) \) to the maximum within the limit set by \( D' \), as described in **ALLOCATION OF RUNNING PACE**.

We confirmed by simulation that, if \( V_1 \) is below \( V_F \), the final race time (i.e., \( t_{\text{tot}} \)) is always longer than \( t_{\text{max}}(5,000) \) (see Fig. 4). In the range of \( V_1 \) below \( V_F \), \( t_{\text{tot}} \) was gradually increased as a function of decreasing \( V_1 \). This effect was, naturally, more pronounced the longer was \( X_1 \). The upper limits of \( V_1 \) are automatically determined by the limit of the best \( V \)-t curve of the subject and each distance curve to \( X_1 \). That is, by definition no one can run at the speed above the intersection of both curves.

If \( t_1 \) was relatively longer for \( X_2 \), the runner would have to run considerably faster in \( X_2 \) as shown in the relationship between \( t_1 \) and \( V_2 \) in Fig. 5 (top left). The dotted line shows the \( V_2 \) required to keep the same \( t_{\text{tot}} \) as \( V_{\text{max}}(X_1) \) and the bold line shows the calculated \( V_2 \). For example, in the specific case of the combination of \( X_1-X_2 \), at the shortest \( t_1 \) (point 1 in Fig. 5, top left), \( V_2 \) is exactly the same as \( V_F \) as a result. This point means that, the runner having run at \( V_{\text{max}}(X_1) \) for the initial \( X_1 \), the required speed for the remaining \( X_2 \) distance is \( V_F \), i.e., the upper (theoretical) limit of “fatigue-free” running speeds (i.e., strategy 1, Fig. 5). If \( t_1 \) took longer, the athlete would have to run faster than \( V_F \) for \( X_2 \), i.e., at \( V_1 \) given by the dotted line (Fig. 5, top left), which is the required \( V_2 \) for the same running time to \( t_{\text{max}}(X) \). However, in the range of \( t_1 \) above the deflection point (point 2 in Fig. 5, top left) of the bold line, the actual calculated \( V_2 \) is dissociated from the required \( V_2 \) (dotted line), due to the “metabolic” limitation. This means that \( V_2 \) can-
not exceed the maximal speed set by the best V-t curve, as determined by the whole D’ and distance X₂ (i.e., strategy 3, Fig. 5, e.g., point 3 in Fig. 5, top left). As a result, this deflection point (point 2 in Fig. 5, top left) means that the athlete must run at V_F for X₁ and at V_max(X₂) for X₂ (i.e., strategy 2, Fig. 5). Therefore, the range between these points (i.e., the range over which t_{tot} can equal to t_{max(X)} represents the maximal possible range for the strategic modification of the race pace.

These conditions are described by the following equations. In strategy 1 (point 1 in Fig. 5, top left), V₁ is V_{max(X)} and V₂ is V_F, t₁ and t₂ are

\[ t_{1(1)} = \left( X_1 - D' \right) / V_F \quad \text{and} \quad t_{2(1)} = X_2 / V_F \quad (15) \]

In the other extreme condition, strategy 2 (point 2 in Fig. 5, top left), V₁ is V_F and V₂ is V_{max(X₂)}, t₁ and t₂ are

\[ t_{1(2)} = X_1 / V_F \quad \text{and} \quad t_{2(2)} = \left( X_2 - D' \right) / V_F \quad (16) \]
The range of \( t_1 \) between these two extreme strategic conditions is
\[
\Delta t_1 = t_{1(2)} - t_{1(1)} = X_1/V_F - (X_1 - D')/V_F = D'/V_F \tag{17}
\]
and also
\[
\Delta t_2 = t_{2(1)} - t_{2(2)} = \Delta t_1 = D'/V_F \tag{18}
\]
This demonstrates an important property of \( D' \) and \( V_F \) on the possible time range (\( \Delta t_1 = \Delta t_2 \)) for a single strategic change of pace. Interestingly, the range is not affected by the total distance, \( X_1 \), or by any combination of \( X_1-X_2 \) or \( V_1-V_2 \); it is purely determined by the runner’s “endurance parameter ratio”: \( D'/V_F \).

Consider also the difference of speeds between two extreme strategic conditions (i.e., conditions 1 and 2)
\[
\Delta V_1 = V_{1(1)} - V_{1(2)} = V_F \cdot [D'(X_1 - D')] \tag{19}
\]
\[
\Delta V_2 = V_{2(2)} - V_{2(1)} = V_F \cdot [D'(X_2 - D')] \tag{20}
\]
where \( X_1 > D' \) and \( X_2 > D' \). This possible speed \( V_2 \) range for the distance \( X_2 \) (\( \Delta V_2 \); \( V_2 \) range) that is permitted for the strategic pace allocation (including the last spurt) is dependent on \( X_2 \).

To examine the effects of \( V_F \) and \( D' \) on the time range and \( V_2 \) range, we plotted the \( t_2-V_2 \) relationship in which \( V_F \) or \( D' \) was changed systematically. The effect of \( D' \) on the \( t_2-V_2 \) relationship under the same \( V_F \) condition, i.e., 5.0 (m/s) is shown in Fig. 6A. As a result, the larger \( D' \) dramatically extended the time range and \( V_2 \) range for the 5,000-m run. However, compared with the effect of \( D' \), the effect of \( V_F \) was relatively small (Fig. 6B). This indicates that \( D' \) is relatively important for the runner’s possible range of pace change, the larger the \( D' \), the larger the potential pace change in the race. \( V_F \) is, of course, an essential determinant of \( t_{\text{max}}(X) \), i.e., the actual level of the race time (Fig. 6B). Therefore, although \( V_F \) seems to be the major determinant of the level of race performance, \( D' \) is an important determinant of the flexibility of race-pace strategy.

### ADDITIONAL REMARKS ON THE ANALYSIS

**P-V Relationship During Running**

Running, either on a treadmill or on a track, provides a major difficulty in terms of assessing the actual power output. However, from the few studies (6–8) that have attempted to determine the actual mechanical work during running, it has been proposed that the efficiency was similar over the range of speeds utilized in this study (i.e., 5–7 m/s), that is, justifying the use of speed as a functional analog of power output. Hughson et al. (4) showed that the \( V-t \) relationship for treadmill running (by using cross-country runners) could be well described by a hyperbolic function at least within the speed range of 5.2–6.3 m/s. The endurance time was 2–12 min, which approximates the range of performance times for middle-distance running (i.e., from 800 to 5,000 m). It was shown that the \( V-t \) curve for the treadmill running, by using the subjects who are not as regularly trained as those of Hughson et al., could also be well fitted the hyperbolic function within the exhaustion time range of 1.9–9.4 min (approximately, within the speed range of 4.2 to 5.5 m/s) (3). Furthermore, the \( V-t \) hyperbolic relationships for treadmill running in subelite middle-distance runners of a university track team were well recognized, with a speed range of \( \sim 4.7–6.5 \) m/s (5). Although there is no experimental evidence that demonstrates that the relationship remains hyperbolic beyond these speed and/or endurance times, the change of speed for the last spurt in actual middle-distance running is likely to be within about \( \pm 1.0 \) m/s. For example, by using the world-class record, the last lap of 52 s in a 3,000-m race means the speed of \( \sim 7.6 \) m/s for the last spurt, compared with the average running speed of \( \sim 6.6 \) m/s (3,000 m/7 min 30 s). We have, therefore, assumed that within this running speed range, the speed will be proportional to the power. It should be recognized that, although we have treated the \( P-t \) curve as if it is hyperbolic for running, this needs further verification. If it proves not to be the case, however, an additional term(s) will need to be added to the characterization (e.g., 13, 19, 23), but this will not alter the conceptual issue being addressed in this study.

**Physiological Interpretation of the Hyperbolic Curve**

The \( P-t \) relationship for high-intensity exercise was described by Hill (1) as early as 1927, and more recently it has been characterized as a hyperbolic function (see Ref. 2 for review). Although the precise physiological determinants of \( \omega_F \) and \( W' \) remain conjectural, the \( \omega_F \) (or CP) has been shown to represent the highest work rate for which a steady state can be attained in pulmonary gas exchange, blood acid-base status, and blood lactate concentration, given sufficient time (14). Therefore, \( \omega_F \) can be regarded as reflecting a rate of energy pool reconstitution that dictates the maximum power that can be sustained without a continued and progressive anaerobic contribution. In many subjects, this highest sustainable lactate level occurs at \( \sim 4 \) meq/l (13–15), although \( \omega_F \) can occur at a wide range of
lactate levels among individual subjects up to 8 meq/l or more, for example (21).

\[ W \]

can be regarded as an energy store composed of \( O_2 \) stores, a phosphagen pool, and a source related to anaerobic glycolysis. It has the units of work and hence represents a constant amount of work that can be performed above \( u_F \), regardless of its rate of performance. \( W \) was not increased by endurance training, whereas \( u_F \) increased systematically; the \( W-t \) curves were well fit by the hyperbola in both conditions (15). Furthermore, Miura et al. (9), recently presented evidence consistent with this view: \( W \) was significantly reduced under glycogen-depleted conditions.

Application Limits of the Hyperbolic Curve

The mechanical limitations of the muscle for high-intensity (i.e., short-duration) exercise is relevant to the condition of \( X < D' \) in our analysis. For such short distances, there is no intersection between the \( V-t \) curve and distance curve to \( X \) as seen in Eqs. 7 and 8. This is consequently beyond the application of the hyperbolic theory. For example, if the runner whose \( V_F \) and \( D' \) are 5.0 (m/s) and 150 (m), respectively, as discussed in GENERAL CONSIDERATIONS, runs 200 m at constant \( V_{max(200)} \), the \( V_{max(200)} \) and \( t_{max(200)} \) are 20 (m/s) and 10 (s), respectively. No athlete can run at such high speed. Therefore, the duration range over which the hyperbolic theory can be applied, might be from 40–50 s to 20–30 min. In other words, these correspond approximately to distances between 400 and 10,000 m of a race for an athlete. Accordingly, we have further assumed in this analysis that the manner in which the race is run (i.e., different pace strategies) does not influence \( D' \) or \( V_F \). Although we know of no experimental evidence to

Fig. 5. Schematic representation of relationship of \( t_1 \) and \( V_2 \) to “t range,” i.e., “permitted” pace allocation for single change of pace. t-range is determined by 2 extreme strategy conditions: 1) \( V_1 = V_{max(X1)} \) and \( V_2 = V_F \) and 2) \( V_1 = V_F \) and \( V_2 = V_{max(X2)} \). If runner chooses strategy 3 (bottom right, left and right; \( V_1 < V_F \) and \( V_2 = V_{max(X2)} \)) intentionally, \( V_2 \) is metabolically limited from achieving required speed (dotted line on top left). Each “strategy,” shown as point nos. (1–3; top left), is represented in graphical detail on right (top to bottom, respectively). Symbols are defined as in Fig. 3.
justify a different assumption, other studies that address this issue could well reduce such an influence; in which case, the theory could be modified accordingly.

One further concern is that this theory would appear to allow the athlete who has “consumed” the entire W’ (or D’) to continue to perform by decreasing the power output to \( \theta_F \) (or \( V_F \)), which represents the upper limit for wholly aerobic energy transfer. However, as the magnitude or intensity of the consequent fatigue is presumably some inverse function of the available W’ (or D’), then the sudden reduction of the power output to \( \theta_F \) (or \( V_F \)) with W’ (or D’) being depleted would, of course, leave the mechanism(s) of fatigue fully expressed. Repleting W’ (or D’) would require a recovery power that is, presumably, below \( \theta_F \) (or \( V_F \)), with the repletion rate likely to be greater than the recovery power below \( \theta_F \) (or \( V_F \)). Further studies are required, however, to determine the intensity-time features of this repletion process.

Characteristics of D’/V_F

As stated in General Considerations, the index D’/V_F (or W’/\theta_F), which we term the endurance parameter ratio, appears to be an important determinant of the race-pace strategy. Here, we further consider the meaning of this index. One might rewrite Eq. 1 as

\[
W’/\theta_F = t \cdot \left( P/\theta_F \right) - 1
\]

Interestingly, the endurance parameter ratio (W’/\theta_F) corresponds to the time for which P/\theta_F is equal to two: the limiting time for an athlete to perform a power output that is twice \( \theta_F \) (i.e., at which W’/\theta_F = 2).

This metabolic relationship is similar to that of chronaxie to rheobase in neurophysiology. If \( \theta_F \) of two competitors are the same, W’ (D’) for the running performance) will determine the flexibility of the intentional race-pace change.

In summary, our results demonstrate that the speed or power at the fatigue threshold (V_F or \( \theta_F \)) and the curvature constant (D’ or W’) parameters of the athlete’s V-t or P-t hyperbolic curve each play an important role in the pace allocation strategy. That is 1) when the running speed during any part of the whole running distance is below V_F, the runner can never attain the goal of achieving the time equivalent to that of running the entire race at constant optimal speed even if the runner attempts to make up for the time lost with a final spurt; and 2) D’ is especially important in determining the flexibility of the race pace that the runner is able to choose intentionally. The ratio of these parameters: D’/V_F (i.e., the endurance parameter ratio) may therefore be considered to be an important determinant of race-pace strategy.

APPENDIX

Theoretical Aspects of General Race-Pace Allocation

In allocation of running pace, we considered only the simple situation in which the runner runs at two different speeds (V_1 and V_2) during two distances (X_1 and X_2) of the total distance X. Here we expand that into more general situations in which the runner runs at n different speeds (V_i) during n distances (X_i) of total distance X. The total distance (X) is divided by n parts (X_i = 1, ..., n). The runner can run at the different speeds V_i in each part X_i, the only assumption being that the total amount of D’ has to be used in the entire race. From a theoretical viewpoint, however, it is sufficient to consider two specific situations, that is, within the limit of D’, during the one part X_i, where V_i is 1) above V_F or 2) below V_F, whereas the runner chooses speed V_i that equals or exceeds V_F [i.e., \( V_F \geq V_i \) for \( V_F \geq V_i \) during the remaining X_i].

**Situation 1:** \( V_F \leq V_i \), \( V_i \geq V_F \), \( V_i \geq V_F \)

This situation means that \( V_i \geq V_F \) excluding \( V_1 \), and \( V_i \geq V_F \) during at least one arbitrary part, X_i. In such a situation, t_i during any part X_i is

\[
t_i = \frac{X_i - D'(X_i)}{V_F}
\]

where D’(X_i) < X_i, and if \( V_i = V_F \) then D’(X_i) = 0. As an assumption

\[
\sum_{i=1}^{n} D'(X_i) = D' = \sum_{i=1}^{n} X_i = X
\]

Therefore, the total time for the race (t_{tot}) is

\[
t_{tot} = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \left|\frac{X_i - D'(X_i)}{V_F}\right| = \left|\frac{1}{V_F}\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} D'(X_i)\right| = \left|\frac{X - D'}{V_F}\right|
\]

Compared with Eq. 8

\[
t_{tot} = t_{max}(X)
\]

That is, as long as the runner keeps the speed \( V_i \) that exceeds \( V_F \) during all X_i, including X_i, by using the whole D’ according to his or her own strategy in the race, the runner can reach...
the goal at the same time to \( t_{\max(x)} \). As a very specific example from the theoretical consideration, if the runner runs at \( V_j \) above \( V_F \) by using the entire \( D' \) during one part, \( X_i \), the runner can reach the goal of achieving the same time as for the constant speed \( V_{\max(x)} \) race, even if the runner chooses every remaining \( V_j \) to be the same as \( V_F \).

Situation 2: \( V_{i+1} \geq V_F, \ V_j < V_F, \ i = 1, \ldots, n \)

This situation means that \( V_j \geq V_F \) excluding \( V_i \), but \( V_j < V_F \) during one part, \( X_i \). As \( V_j \) is below \( V_F \)

\[
\Delta V = V_F - V_j > 0
\]

(26)

and \( t_j \) is

\[
t_j = X_j/V_j = X_j/(V_F - \Delta V)
\]

(27)

In each remaining part \( i \) (i.e., \( i = 1, \ldots, n, i \neq j \)), as \( V_{i+1} \) is above \( V_F \)

\[
t_{i+1} = \left| X_{i+1} - D'[x_{i+1}]/V_F \right|
\]

(28)

where \( D'[x_{i+1}] < X_{i+1} \). Therefore, total running time \( t_{\text{tot}} \) is

\[
t_{\text{tot}} = t_j + \sum_{k=1}^{n} t_k = X_j/V_F - \Delta V + \left| 1/V_F \right| \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} D'[x_k]
\]

(29)

where \( k = 1, \ldots, n \), \( k \neq j \), and \( D' = \sum_{k=1}^{n} D'[x_k] \). Time difference between \( t_{\text{tot}} \) and \( t_{\max(x)} \)

\[
\Delta t = t_{\text{tot}} - t_{\max(x)} = X_j/(V_F - \Delta V) + \left| 1/V_F \right| \sum_{k=1}^{n} X_k - D'
\]

(30a)

\[
- \left( 1/V_F \right) \left| X - D' \right| = X_j/\left| 1/V_F - \Delta V \right| - \left( 1/V_F \right)
\]

From Eq. 26, \( \Delta V > 0 \), so \( V_j < 1/(V_F - \Delta V) \); therefore

\[
\Delta t > 0
\]

(30b)

As a consequence, if a runner runs at a speed \( V_j \) which is less than \( V_F \) during any one part of the race, \( X_i \) (i.e., \( V_j \) in \( X_i \)), the "lost" time cannot be made up during the remaining parts, even if the runner chooses all remaining \( V_j \) that are above \( V_F \) (but within the limit of his or own \( D' \)). That is, the runner can never attain the goal of achieving the time equivalent to that of running the entire race at the constant \( V_{\max(x)} \).

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